Packing graphs of bounded codegree

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Overview

- What is graph packing, what is known?
- New conditions under which two graphs pack, with respect to the Bollobás-Eldridge-Catlin conjecture.
- Rough proof sketch with pictures.

Graph packing: definition

Graphs G_1, G_2 on n vertices.

Definition

G_1 and G_2 are said to *pack* if there exist injective functions (labellings) of their vertex sets into $\{1, 2, \ldots, n\}$ such that their edge sets have disjoint images.

Equivalently:

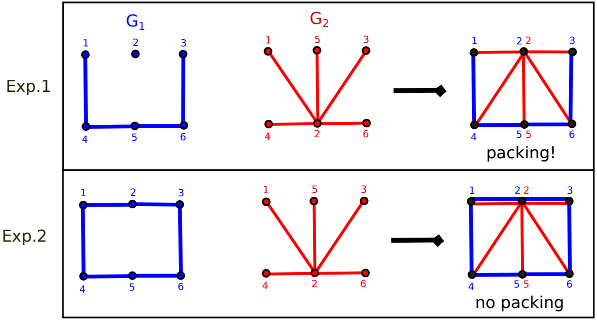
Definition

G_1 and G_2 are said to *pack* if G_1 is a subgraph of $\overline{G_2}$.

Graph packing: examples

Colour convention

- From now on, G_1 is 'blue' and G_2 is 'red'.
- Given a labelling, a (multi-)edge that is red and blue is called purple.



Note: G_1 and G_2 pack iff there exists a labelling without purple edges.

The BEC conjecture

Let $\Delta_i = \Delta(G_i)$ be the maximum degree of G_i.

BEC conjecture [Bollobás, Eldridge, 1978 and Catlin, 1976]

Graphs G_1 and G_2 on n vertices pack if

$$(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1.$$

- *Sharp* if true, because we can take G_1 and G_2 to be two collections of disjoint cliques that satisfy $(\Delta_1 + 1)(\Delta_2 + 1) = n + 2$ and do not pack.
- Would generalize Hajnal-Szmerédi theorem.

Some known results

Recall BEC condition: $(\Delta_1+1)(\Delta_2+1) \leq n+1$.

There exists a packing if one of the following holds

- $2\Delta_1\Delta_2 < n$; [Sauer,Spencer, 1978]
- $(\Delta_1+1)(\Delta_2+1) \leq 0.6 \cdot n + 1$; [Kaul, Kostochka, Yu, 2008]
- BEC and $\Delta_1 \leq 2$; [Aigner, Brandt, 1993]
- BEC and $\Delta_1 = 3$ and n huge ; [Csaba, Szemerédi,Shokoufandeh, 2003]
- BEC and one of the graphs is bipartite; [Csaba, 2003]
- BEC and one of the graphs is random (w.h.p.). [Bollóbas, Janson, Scott, 2014+]

Furthermore,

Under BEC condition, there exists a *near-packing* of degree 1. [Eaton, 2000]

New results (1)

Definition

$K_{2,t}$ denotes the complete bipartite graph on 2 and t vertices.

The following is a corollary of our somewhat more technical main theorem.

Corollary [CvB, Kang, 2016+]

Given an integer $t \geq 2$, the BEC conjecture holds under the additional condition that $\Delta_1 > 20t \cdot \Delta_2$ and $K_{2,t}$ is *not* a subgraph of G_1.

New results (2)

Theorem [CvB, Kang, in preparation]

The BEC conjecture holds under the additional condition that $\Delta_1 > 10^7$ and neither G_1 nor G_2 contains a cycle of length $4, 6$ or 8 as a subgraph.

Connection to equitable colourings

Definition

An equitable colouring is a proper vertex colouring such that the sizes of the colour classes pairwise differ at most 1.

Corollary [CvB, K, 2016+]

Every $K_{2,t}-$free graph ($t \geq 2$) on n vertices with maximum degree $\Delta \geq \sqrt{20t} \cdot \sqrt{n}$ has an equitable $\Delta-$colouring.

[This result cannot be obtained by the Hajnal-Szmerédi theorem.]

Flavour of the proofs.

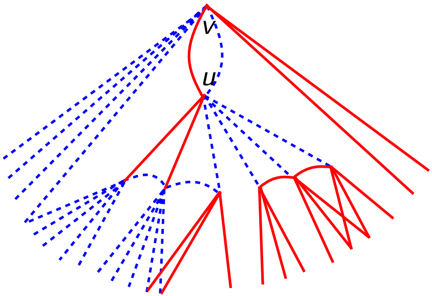
We sketch a proof of the Sauer+ Spencer result from 1978, that $2\Delta_1\Delta_2 + 1 < n$ is sufficient for packing. Consider hypothetical counterexamples that satisfy this inequality but for which the graphs do not pack.

Claim 1

Every edge-minimal counterexample (MCE) has exactly 1 purple edge.

Claim 2

Given a MCE with purple edge uv, *every* vertex (except possibly v) is connected to u with a red–blue-link or a blue–red-link.



Therefore

$$\begin{aligned} n &\leq |\text{blue–red-n'hood of } u| + |\text{red–blue-n'hood of } u| + |\{v\}| \\ &\leq \Delta_1\Delta_2 + \Delta_2\Delta_1 + 1. \end{aligned}$$

Contradiction. □

Proof strategy

To prove our main theorem we also consider a MCE. A case distinction is made and for each case we derive a contradictory upper bound on n. This involves identifying and comparing many vertex subsets and eventually deriving that the sum of their sizes is small, using:

- It is impossible to eliminate the purple edge by rearranging labels (otherwise $\exists$ packing).
- $K_{2,t}$ is not a subgraph of G_1, so blue neighbourhoods of distinct vertices cannot have too much overlap.
- The degrees of G_1 and G_2 are bounded.

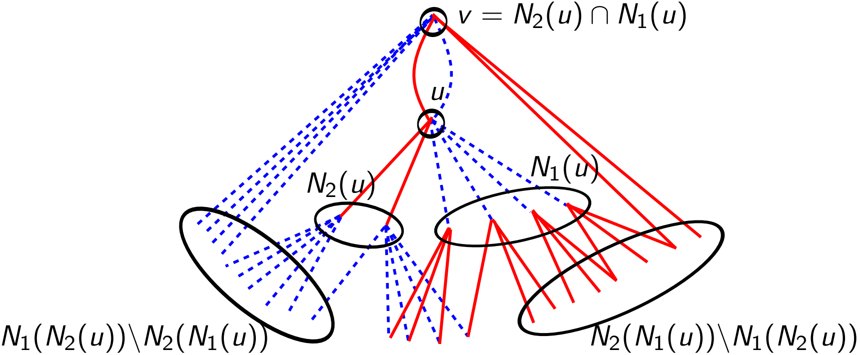


Figure : The basic neighbourhood structure of a hypothetical critical counterexample, as seen from u.

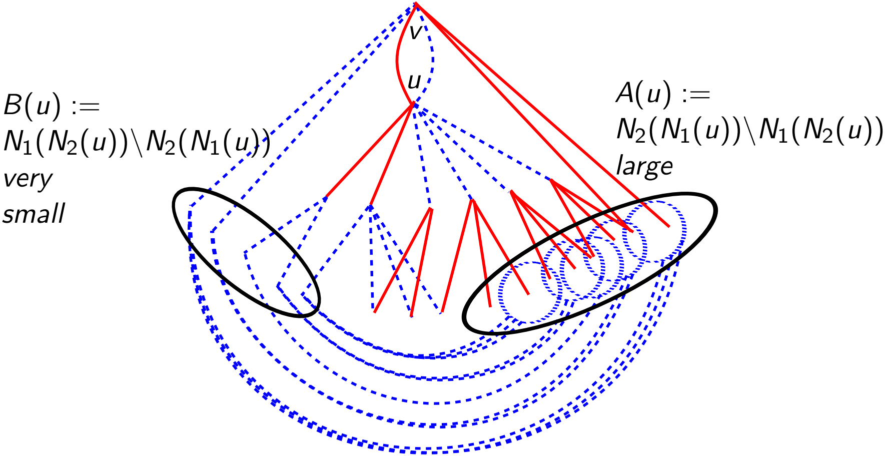


Figure : If $|A(u)| = \Omega(\Delta_2^2)$ then $|B(u)| = O(\Delta_2)$, so $n = \Delta_1\Delta_2 + O(\Delta_2)$.

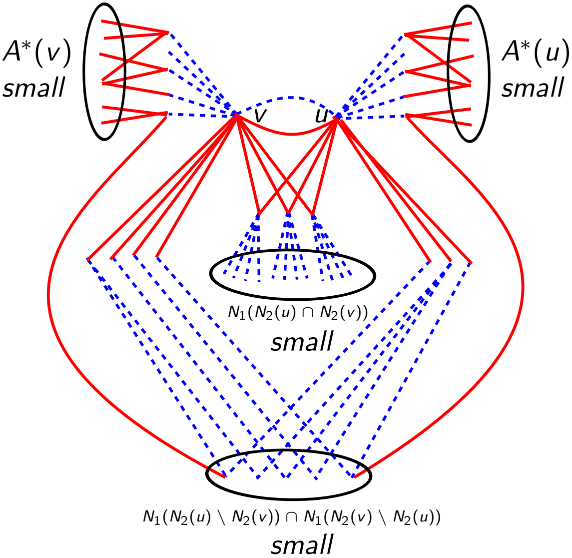


Figure : Each of the encircled sets is small.

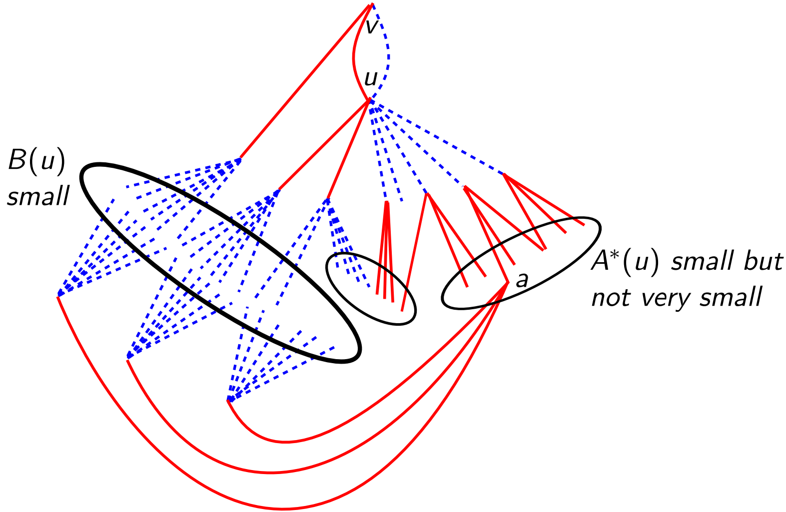


Figure : The red neighbourhoods of a and u are disjoint, for some $a \in A^*(u)$. The common blue neighbourhood of these red neighbourhoods cannot be too large, or else there exists a blue copy of $K_{2,t}$.

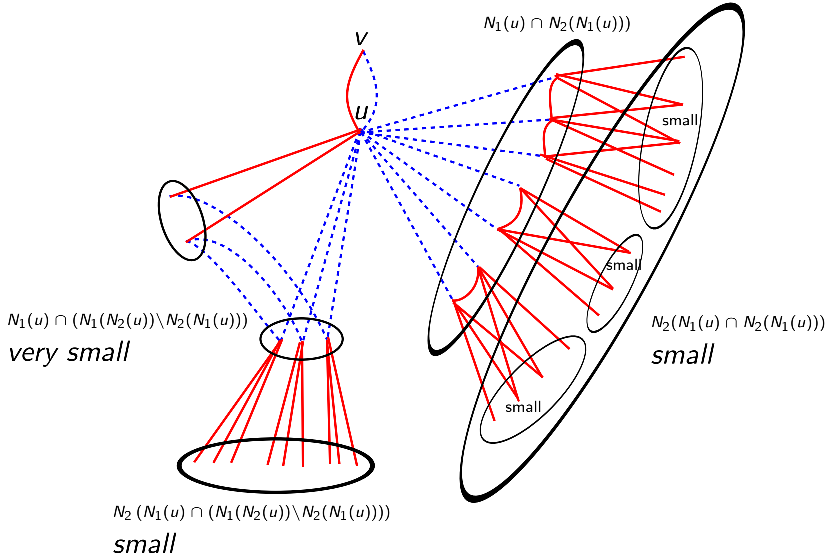


Figure : Case distinction: can assume that the common red neighbourhoods of red-adjacent vertices have a large overlap, so the union of such red neighbourhods is small.

Concluding remarks

- The BEC conjecture on graph packing holds under each of the following sets of additional conditions.
  - G_1 does not contain $K_{2,t}$ as a subgraph, and $\Delta_1 > 20t \cdot \Delta_2$.
  - G_1 and G_2 do not contain C_4, C_6 or C_8 as a subgraph, and $\Delta_1 > 10^7$.
- The proofs go by comparing (sizes of) vertex subsets in a minimal counterexample and deriving (nested) contradictions.
- What about other sparsity conditions?
  - Excluding triangles.
  - Excluding several larger cycles.

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